Exact propagator for generalized Ornstein-Uhlenbeck processes

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A closed form expression for the propagator is derived, in terms of modified Bessel functions, for the Fokker-Planck equation for a physically important generalization of the Ornstein-Uhlenbeck process where the diffusion constant $D(p)$ is a function of the momentum. The closed form is found for the general case $D(p)\sim|p|^{-\alpha}$ where $\alpha\geqslant 0$ and leads to the standard Gaussian form for $\alpha=0$. The propagator for the specific case $D(p)\sim|p|^{-1}$ is used to derive analytic expressions for probability distributions and correlation coefficients. An exact expression is found for the constant of proportionality for the anomalous diffusion of the mean-square displacement of a particle at short times.

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I. INTRODUCTION

The Ornstein-Uhlenbeck process [1] plays a central role in the theory of diffusion where a particle of mass m and momentum p is subject to a drag force $-\gamma p$, where γ is a constant, and a random rapidly varying force $f(t)$. This process has been generalized (see [2,3]) to include random forces $f(x,t)$ which depend on both the position of the particle x and the time t. The equation of motion for such a generalized Ornstein-Uhlenbeck process can be written as

$$\dot{p}=-\gamma p+f(x,t). \tag{1}$$

We assume the force $f(x,t)$ has the following statistics:

$$\langle f(x,t)\rangle=0,\quad \langle f(x,t)f(x',t')\rangle=C(x-x',t-t'), \tag{2}$$

where the angular brackets indicate an average over an ensemble of particles and C denotes a space-time correlation function with spatial and temporal correlations lengths ℓ and τ, respectively. In the limit where τ approaches zero and for weak damping γ the equation of motion can be approximated by a Langevin equation. Considering the ensemble average of the first and second moments of dp one can show, in the usual way, that this stochastic system leads to a Fokker-Planck equation or a generalized diffusion equation for the probability density of the momentum p at time t, $P(p,t)$, in which the momentum diffusion constant $D(p)$ depends explicitly on the momentum [3], i.e.,

$$D(p)=\frac{1}{2}\int_{-\infty}^{\infty}C(pt/m,t)dt. \tag{3}$$

The general form of the Fokker-Planck equation for P is then

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial p}\left(\gamma p+D(p)\frac{\partial}{\partial p}\right)P. \tag{4}$$

Sturrock [4] was the first to derive a Fokker-Planck equation with a diffusion constant that depends on the momentum p as in Eq. (4). He analyzed the motion of a charged particle in a uniform magnetic field subject to randomly varying weak electric fields and no damping, i.e., $\gamma=0$, in above. He showed that it was possible to describe this stochastic acceleration process completely in terms of a correlation function of the general form given in Eq. (2). Golubovic *et al.* [5,6] considered the motion of a particle, also with $\gamma=0$, subject to a random force as in Eq. (1), but which can be written as the gradient of a potential, the random potential having statistics represented by Eq. (2), where C is taken to be of Gaussian form. They evaluated $D(p)$ and showed that $D(p)\sim|p|^{-3}$ for $p\gg p_0$ where $p_0=m\ell/\tau$. Arvedson and co-workers [2,3] considered the case where for most of the time the momentum p is large compared to p_0. They analyzed the Fokker-Planck equation for a generic random force in Eq. (1) with damping and showed by expanding Eq. (3) for large momentum that the Fokker-Planck equation has a diffusion constant $D(p)\sim|p|^{-1}$. When the force can be written as the gradient of a potential one recovers $D(p)\sim|p|^{-3}$. Fokker-Planck equations with momentum dependent diffusion constants have also been applied to study stochastic acceleration of particles in plasmas.

Recently [2,3,7] the solution of the Fokker-Planck equation (4) was considered for nonzero γ and for both the specific case of $D(p)=D_0|p|^{-1}$, where D_0 is a constant, and the general case $D(p)\sim|p|^{-\alpha}$ for $\alpha\geqslant 0$. Note that setting $\alpha=0$ gives the standard Ornstein-Uhlenbeck process. For clarity we will initially consider the specific case $D(p)=D_0|p|^{-1}$. The corresponding Fokker-Planck equation (4) can be written in scaled variables $t'=\gamma t$ and $z=(\gamma/D_0)^{1/3}p$, as

$$\frac{\partial P}{\partial t'}=\frac{\partial}{\partial z}\left(z+\frac{1}{|z|}\frac{\partial}{\partial z}\right)P\equiv\hat{F}P. \tag{5}$$

The general solution of the Fokker-Planck equation (5) can be written in terms of the propagator for the equation $K(y,z,t')$, which is the probability density for the scaled momentum to reach z after a time t' starting from y at time $t'=0$, i.e., $K(y,z,0)=\delta(z-y)$. Arvedson *et al.* [2] calculated this propagator by spectral decomposition [8], i.e., by first transforming the Fokker-Planck operator $\hat{F}$ to a Hermitian form using the stationary solution to the equation $P_0(z)\propto\exp(-|z|^3/3)$,

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$$\hat{H} = P_0^{-1/2}\hat{F}P_0^{1/2} = \frac{d}{dz}\frac{1}{|z|}\frac{d}{dz} + \frac{1}{2} - \frac{|z|^3}{4}. \tag{6}$$

The propagator can then be expressed as a sum involving the eigenfunctions and eigenvalues of $\hat{H}$, and probability distributions, moments, and correlation coefficients can also be expressed in terms of sums of matrix elements. Arvedson *et al.* [2] used an unusual set of creation and annihilation operators to show that the eigenvalues of $\hat{H}$ formed a set of staggered ladder spectra. Without explicitly constructing the eigenfunctions they showed that they could calculate correlations coefficients by evaluating matrix elements in a rather involved way using the creation and annihilation operators. In a recent paper [7] we presented an alternate solution to the eigenvalue equation for $\hat{H}$ and in so doing we obtained explicit expressions for the eigenfunctions in terms of associated Laguerre polynomials. The matrix elements involved in the sums for calculating the correlation coefficients could thus be evaluated in a more straightforward way using the recurrence relations and other properties of the associated Laguerre polynomials. We now show here how we can use these eigenvalues and eigenfunctions to obtain a closed-form expression for the propagator K of the Fokker-Planck equation (5) which gives a closed-form solution to the problem. Knowing the propagator greatly facilitates the calculation of physical quantities such as correlation coefficients. We show how probability distributions and correlation coefficients can be calculated directly by integration without the need for infinite sums over matrix elements. We also obtain an exact expression for the constant of proportionality describing the short-time anomalous diffusion dynamics for the mean-squared position of the particle. In addition we give a closed-form expression for the propagator for the general case where $D(p) \sim |p|^{-\alpha}$ for $\alpha \geqslant 0$ and show that this reduces to the well-known Gaussian form for $\alpha = 0$ [1,8].

The layout of the paper is as follows: In Sec. II we summarize the known results for the solution of the eigenvalue problem. In Sec. III we show how the eigenvalues and eigenfunctions can be used to obtain a closed-form expression for the propagator. In Sec. IV we show how the correlation functions are calculated and in Sec. V the propagator for the general case is given. In Sec. VI we give a summary of our results.

II. EIGENVALUES AND EIGENFUNCTIONS

The eigenvalue problem for the operator $\hat{H}$ is

$$\frac{d}{dz}\frac{1}{|z|}\frac{d\psi}{dz} + \left(\frac{1}{2} - \frac{|z|^3}{4}\right)\psi = \lambda\psi \tag{7}$$

with $\psi \to 0$ as $|z| \to \infty$. The operator $\hat{H}$ commutes with the parity operator and so the eigenfunctions can be divided into even or odd with respect to $z \to -z$.

As found in [2,7] the even eigenvalues are

$$\lambda_n^+ = -3n, \quad n = 0,1,2\ldots \tag{8}$$

and the odd eigenvalues are given by

$$\lambda_n^- = -3n - 2, \quad n = 0,1,2\ldots\ . \tag{9}$$

The even and odd eigenvalues separately are evenly spaced like those of a harmonic oscillator but they are shifted relative to one another giving the staggered ladder spectra found in [2].

The eigenfunctions were derived in [7] and the normalized even eigenfunctions were found to be

$$\psi_n^+ = 3^{1/3}\sqrt{\frac{1}{2}\frac{\Gamma(n+1)}{\Gamma(n+1/3)}}L_n^{-2/3}\left(\frac{|z|^3}{3}\right)\exp\left(-\frac{|z|^3}{6}\right), \tag{10}$$

where $L_n^{-2/3}$ are associated Laguerre polynomials and $n = 0,1,2\ldots$.

Similarly the normalized odd eigenfunctions are

$$\psi_n^- = 3^{-1/3}\sqrt{\frac{1}{2}\frac{\Gamma(n+1)}{\Gamma(n+5/3)}}z|z|L_n^{2/3}\left(\frac{|z|^3}{3}\right)\exp\left(-\frac{|z|^3}{6}\right) \tag{11}$$

with $n = 0,1,2\ldots$.

III. PROPAGATOR

The general solution of the Fokker-Planck equation (5) can be written in terms of the propagator K. The propagator $K(y,z,t')$ with the initial condition $K(y,z,0) = \delta(z-y)$ can be expanded in terms of the eigenvalues and eigenfunctions of the operator $\hat{H}$ above as

$$K(y,z,t') = \sum_{n\sigma} P_0^{-1/2}(y)\psi_n^\sigma(y)P_0^{1/2}(z)\psi_n^\sigma(z)e^{\lambda_n^\sigma t'}, \tag{12}$$

where $\sigma = \pm$ for the even and odd eigenfunctions [3,8]. Since $P_0^{1/2}(y) = \psi_0^+(y)$, we can write K explicitly in terms of the even and odd eigenfunctions given in Sec. II as

$$K(y,z,t') = \frac{\psi_0^+(z)}{\psi_0^+(y)} \times \left(\sum_{n=0}^{\infty}\psi_n^+(y)\psi_n^+(z)e^{\lambda_n^+ t'} + \sum_{n=0}^{\infty}\psi_n^-(y)\psi_n^-(z)e^{\lambda_n^- t'}\right). \tag{13}$$

Substituting the eigenvalues from Eqs. (8) and (9) and the eigenfunction expressions from Eqs. (10) and (11) we get

$$K(y,z,t') = e^{-|z|^3/3}\left[\frac{3^{2/3}}{2}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/3)}L_n^{-2/3}(|y|^3/3)L_n^{-2/3}(|z|^3/3)e^{-3nt'} + \frac{e^{-2t'}yz|yz|}{2.3^{2/3}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+5/3)}L_n^{2/3}(|y|^3/3)L_n^{2/3}(|z|^3/3)e^{-3nt'}\right]. \tag{14}$$

Since we have explicit expressions for the eigenfunctions in terms of associated Laguerre polynomials the infinite summations can be evaluated exactly from the following expression given in [9]:

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\beta+1)}L_n^{\beta}(y)L_n^{\beta}(z)\delta^n = \frac{(yz\delta)^{-\beta/2}}{1-\delta}\exp\left[-\delta\left(\frac{y+z}{1-\delta}\right)\right]I_\beta\left(\frac{2\sqrt{yz\delta}}{1-\delta}\right) \quad \text{for } |\delta|<1, \tag{15}$$

where I_β is a modified Bessel function [10]. Taking $\delta = e^{-3t'}$ with $t'>0$ we get

$$K(y,z,t') = a\exp\left[-b|y|^3 - \left(b+\frac{1}{3}\right)|z|^3\right] \times [|yz|I_{-2/3}(c|yz|^{2/3}) + yzI_{2/3}(c|yz|^{2/3})], \tag{16}$$

where

$$a = e^{-t'}/2(1-e^{-3t'}), \quad b = e^{-3t'}/3(1-e^{-3t'}),$$

$$c = 2e^{-3t'/2}/3(1-e^{-3t'}). \tag{17}$$

All quantities of interest for generalized Ornstein-Uhlenbeck processes can be evaluated from this closed-form expression for the propagator. For example, the probability distribution at any time t' is given in terms of the distribution at $t'=0$ by

$$P(z,t') = \int_{-\infty}^{\infty} dyK(y,z,t')P(y,0). \tag{18}$$

In particular if the particle is initially at rest, i.e., with initial momentum distribution $P(y,0)=\delta(y)$, we get from Eq. (18) that $P(z,t')=K(0,z,t')$. So taking the limit as $y\to 0$ in Eq. (16) and using the fact that $I_\alpha(z)\sim\frac{1}{\Gamma(\alpha+1)}\left(\frac{1}{2}z\right)^\alpha$ as $z\to 0$, we get immediately

$$P(z,t') = K(0,z,t') = \frac{3}{2\Gamma(1/3)}\frac{\exp[-|z|^3/3(1-e^{-3t'})]}{[3(1-e^{-3t'})]^{1/3}} \tag{19}$$

or expressing this in the original nonscaled variables p and t,

$$P(p,t) = \frac{1}{2\Gamma(4/3)}\frac{\gamma^{1/3}}{[3D_0(1-e^{-3\gamma t})]^{1/3}}\exp\left[-\frac{\gamma|p|^3}{3D_0(1-e^{-3\gamma t})}\right]. \tag{20}$$

This expression was proposed as an ansatz in [3] but we see here that it emerges naturally from the closed-form expression for K.

IV. CORRELATION FUNCTIONS AND DIFFUSION AT SHORT TIMES

We now use K in Eq. (16) to show how correlation coefficients can be determined directly without the need to evaluate infinite sums over matrix elements. We also give an analytic expression for the constant of proportionality describing the short-time anomalous diffusion dynamics for the mean-squared position of the particle.

The equilibrium or stationary correlation coefficients for an observable $O(z)$ can be written as [3,8]

$$\langle O(z_{t'})O(z_0)\rangle_{eq.} = \int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dyO(z)O(y)K(y,z,t')P_0(y). \tag{21}$$

The equilibrium momentum correlation coefficient is then

$$\langle z_{t'}z_0\rangle_{eq.} = \int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dyzyK(y,z,t')P_0(y). \tag{22}$$

Substituting Eq. (16) into the above, the integral over the term involving $I_{-2/3}$ vanishes since the integrand is odd in y so we get

$$\langle z_{t'}z_0\rangle_{eq.} = \frac{a3^{2/3}}{2\Gamma(1/3)}\int_{-\infty}^{\infty}dzz^2e^{-(b+1/3)|z|^3} \times \int_{-\infty}^{\infty}dyy^2e^{-(b+1/3)|y|^3}I_{2/3}(c|z|^{3/2}|y|^{3/2}). \tag{23}$$

Substituting $u=y^3$ we obtain

$$\langle z_{t'}z_0\rangle_{eq.} = \frac{a3^{-1/3}}{\Gamma(1/3)}\int_{-\infty}^{\infty}dzz^2e^{-(b+1/3)|z|^3}\int_0^{\infty}due^{-(b+1/3)u}I_{2/3}(c|z|^{3/2}\sqrt{u}). \tag{24}$$

The integral over u can then be calculated analytically [9], giving

$$\langle z_{t'} z_0 \rangle = \frac{3^{-4/3}4}{\Gamma(5/3)} \frac{a}{c\sqrt{b+1/3}} \times \int_0^\infty dz z^{1/2} \exp\{-[b+1/3-c^2/8(b+1/3)]z^3\} \times M_{-1/2,1/3}\left(\frac{c^2 z^3}{4(b+1/3)}\right), \quad (25)$$

where M is a Whittaker function.

If we put $v = z^3$ we get

$$\langle z_{t'} z_0 \rangle = \frac{3^{-7/3}4}{\Gamma(5/3)} \frac{a}{c\sqrt{b+1/3}} \times \int_0^\infty dv v^{-1/2} \exp\{-[b+1/3-c^2/8(b+1/3)]v\} \times M_{-1/2,1/3}\left(\frac{c^2 v}{4(b+1/3)}\right). \quad (26)$$

This integral can also be done analytically [9] using the fact that

$$\int_0^\infty dx e^{-sx} x^\alpha M_{\mu,\nu}(x) = \frac{\Gamma(\alpha+\nu+3/2)}{(1/2+s)^{\alpha+\nu+3/2}} \times {}_2F_1[\alpha+\nu+3/2, -\mu+\nu+1/2, 2\nu+1, 2/(2s+1)], \quad (27)$$

where ${}_2F_1$ is a hypergeometric function [10]. Substituting for a, b, and c from Eq. (17) into the result we finally get

$$\langle z_{t'} z_0 \rangle_{eq.} = \frac{3^{-1/3}\Gamma(4/3)}{\Gamma(5/3)} e^{-2t'} {}_2F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, e^{-3t'}\right) \quad (28)$$

or, in the original unscaled variables,

$$\langle p_t p_0 \rangle_{eq.} = \frac{3^{-1/3}\Gamma(4/3)}{\Gamma(5/3)} \left(\frac{D_0}{\gamma}\right)^{2/3} e^{-2\gamma t} {}_2F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, e^{-3\gamma t}\right) \quad (29)$$

which is here obtained directly without the need to evaluate a set of matrix elements and then perform an infinite sum [3,7].

The time dependence for the mean-square displacement for a particle initially at rest at the origin is given in terms of the momentum correlation function [3,8] by

$$\langle x^2(t) \rangle = \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \langle p_{t_2} p_{t_1} \rangle = \frac{1}{\gamma^2} \left(\frac{D_0}{\gamma}\right)^{2/3} \int_0^{t'} dt_1' \int_0^{t'} dt_2' \langle z_{t_2'} z_{t_1'} \rangle, \quad (30)$$

where

$$\langle z_{t_2'} z_{t_1'} \rangle = \int_{-\infty}^\infty dz_1 \int_{-\infty}^\infty dz_2 z_1 z_2 P(z_2, t_2'; z_1, t_1') = \int_{-\infty}^\infty dz_1 \int_{-\infty}^\infty dz_2 z_1 z_2 K(z_1, z_2, t_2' - t_1') K(0, z_1, t_1'). \quad (31)$$

Substituting the expressions for the $K(z_1, z_2, t_2' - t_1')$ from Eq. (16) with $t_2' > t_1' > 0$ and $K(0, z_1, t_1')$ from Eq. (19) we get the same type of integrals as used in the evaluation of $\langle z_{t'} z_0 \rangle_{eq.}$ above. These can be performed analytically and the final result is

$$\langle z_{t_2'} z_{t_1'} \rangle = \frac{3^{-1/3}\Gamma(4/3)}{\Gamma(5/3)} \frac{e^{-2(t_2'-t_1')}(1-e^{-3t_1'})}{(1-e^{-3t_2'})^{1/3}} \times {}_2F_1\left[\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, e^{-3(t_2'-t_1')}\left(\frac{1-e^{-3t_1'}}{1-e^{-3t_2'}}\right)\right]. \quad (32)$$

For short times the expression in Eq. (32) can be expanded in powers of t_1' and t_2' about the origin (using a symmetric expression for $t_1' > t_2'$) and thus the double integral in t_1' and t_2' in Eq. (30) can be performed analytically. Writing the result in terms of the unscaled variables we get

$$\langle x^2(t) \rangle = 3^{-2/3} \frac{47}{40} \frac{\Gamma(4/3)}{\Gamma(5/3)} \frac{(D_0)^{2/3}}{m^2} t^{8/3} + O(t^{11/3}). \quad (33)$$

The actual value of the constant $3^{-2/3}\frac{47}{40}\frac{\Gamma(4/3)}{\Gamma(5/3)}$ is 0.5588. This proportionality constant was evaluated numerically in [2] by approximating various integrals and infinite summations giving a value of 0.57. Using the propagator one can obtain the exact value simply without approximations.

V. PROPAGATOR FOR THE GENERAL CASE

The above approach can be generalized in a straightforward way to the case when the momentum diffusion constant in Eq. (4), $D(p)$, has a more general form $D(p) \sim |p|^{-\alpha}$ where $\alpha \geqslant 0$. After transforming the Fokker-Planck equation to its Hermitian form, the eigenvalue problem in the general case becomes

$$\frac{d}{dz} \frac{1}{|z|^\alpha} \frac{d\psi}{dz} + \left(\frac{1}{2} - \frac{|z|^{2+\alpha}}{4}\right)\psi = \lambda \psi. \quad (34)$$

As found in [3,7] the even eigenvalues are given by

$$\lambda_n^+ = -(2+\alpha)n, \quad n = 0, 1, 2 \ldots \quad (35)$$

and the odd eigenvalues by

$$\lambda_n^- = -(2+\alpha)n - \alpha - 1, \quad n = 0, 1, 2 \ldots . \quad (36)$$

The eigenfunctions were derived in [7] and can be written in terms of associated Laguerre polynomials giving the normalized even eigenfunctions as

$$\psi_n^+(z) = (\alpha+2)^{(\alpha+1)/(2\alpha+4)} \sqrt{\frac{1}{2}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{\alpha+2}\right)}} \times L_n^{-(\alpha+1)/(\alpha+2)}\left(\frac{|z|^{\alpha+2}}{\alpha+2}\right)\exp\left(-\frac{|z|^{\alpha+2}}{2(\alpha+2)}\right) \quad (37)$$

for $n=0,1,2\ldots$.

Similarly the normalized odd eigenfunctions are

$$\psi_n^-(z) = (\alpha+2)^{-(\alpha+1)/(2\alpha+4)} \times \sqrt{\frac{1}{2}\frac{\Gamma(n+1)}{\Gamma\left(n+2-\frac{1}{\alpha+2}\right)}}z|z|^{\alpha} \times L_n^{(\alpha+1)/(\alpha+2)}\left(\frac{|z|^{\alpha+2}}{\alpha+2}\right)\exp\left(-\frac{|z|^{\alpha+2}}{2(\alpha+2)}\right) \quad (38)$$

with $n=0,1,2\ldots$.

Using Eqs. (13) and (15) a closed-form expression for the propagator can be written as

$$K(y,z,t') = a' \exp\left(-b'|y|^{\alpha+2} - \left(b+\frac{1}{\alpha+2}\right)|z|^{\alpha+2}\right) \times [|yz|^{(\alpha+1)/2}I_{-(\alpha+1)/(\alpha+2)}(c'|yz|^{(\alpha+2)/2}) + yz|yz|^{(\alpha-1)/2}I_{(\alpha+1)/(\alpha+2)}(c'|yz|^{(\alpha+2)/2})], \quad (39)$$

where

$$a' = e^{-(\alpha+1)t'/2}/2(1-e^{-(\alpha+2)t'}),$$

$$b' = e^{-(\alpha+2)t'}/(\alpha+2)(1-e^{-(\alpha+2)t'}),$$

$$c' = 2e^{-(\alpha+2)t'/2}/(\alpha+2)(1-e^{-(\alpha+2)t'}). \quad (40)$$

Similar results to those derived in Sec. III can be obtained for the general case using the propagator in Eq. (39), in particular for the case $\alpha=3$ considered by Golubovic *et al.* [5,6].

Note also that if we take $\alpha=0$ in Eqs. (39) and (40) the modified Bessel functions become $I_{-1/2}$ and $I_{1/2}$ which can be written in terms of a hyperbolic cosine and sine, respectively [10], and the propagator K reduces to

$$K(y,z,t') = \frac{1}{[2\pi(1-e^{-2t'})]^{1/2}} \exp\left[-\frac{(z-ye^{-t'})^2}{2(1-e^{-2t'})}\right], \quad (41)$$

i.e., we recover the classic Gaussian form for the standard Ornstein-Uhlenbeck process [1,8].

VI. SUMMARY

We have derived a closed-form expression for the propagator for generalized Ornstein-Uhlenbeck processes, where the diffusion constant is an explicit function of the momentum. The propagator for the general case $D(p)\sim|p|^{-\alpha}$ where $\alpha\geqslant 0$ is given and reduces to the usual Gaussian form for $\alpha=0$. Knowing a closed form for the propagator facilitates calculations of physical interest by allowing one to write down integral expressions for probability distributions and correlation coefficients. For the specific case $D(p)\sim|p|^{-1}$ we have shown how correlation coefficients can be calculated analytically using the closed form for K. We have also shown how to evaluate the exact form of the proportionality constant for anomalous diffusion in the mean-square displacement at short times.

[1] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).

[2] E. Arvedson, M. Wilkinson, B. Mehlig, and K. Nakamura, Phys. Rev. Lett. **96**, 030601 (2006).

[3] V. Bezuglyy, B. Mehlig, M. Wilkinson, K. Nakamura, and E. Arvedson, J. Math. Phys. **47**, 073301 (2006).

[4] P. A. Sturrock, Phys. Rev. **141**, 186 (1966).

[5] L. Golubovic, S. Feng, and F.-A. Zeng, Phys. Rev. Lett. **67**, 2115 (1991).

[6] M. N. Rosenbluth, Phys. Rev. Lett. **69**, 1831 (1992).

[7] F. Mota-Furtado and P. F. O'Mahony, Phys. Rev. A **74**, 044102 (2006).

[8] C. W. Gardiner, *Handbook of Stochastic Methods*, 3rd Ed. (Springer, Berlin, 2004).

[9] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic Press, New York, 1965).

[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).